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SPORTING CONTESTS WITH GAMES ON THE FIELD

Concursos deportivos con juegos en el campo

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ABSTRACT: This article studies professional sports leagues where each team consists of offensive and defensive units and plays games on the field during the league's regular season. It is shown that (i) if each team independently determines the offensive and defensive talent levels demanded so as to maximize the expected win percentage (Pythagorean expectation) subject to the budget constraint, then the teams in the leagues are divided into rich offensive-minded winners and poor defensive-minded losers, and that (ii) if the payroll of each team is the same, then the unique equilibrium talent allocation is Pareto efficient. Otherwise, it is Pareto inefficient.

KEY WORDS: sports league; games on the field; win maximization; Pythagorean expectation; team payroll; Pareto efficiency.

RESUMEN: *Este artículo estudia las ligas deportivas profesionales donde cada equipo consta de unidades ofensivas y defensivas y juega partidos en el campo durante la temporada regular de la liga. Se muestra que (i) si cada equipo determina independientemente los niveles de talento ofensivo y defensivo exigidos para maximizar el porcentaje de victorias esperadas (expectativa Pitagórica) sujeto a la restricción presupuestaria, entonces los equipos de las ligas se dividen en ricos con mentalidad ofensiva ganadores y malos perdedores de mentalidad defensiva, y que (ii) si la nómina de cada equipo es la misma, entonces la asignación de talento de equilibrio único es Pareto eficiente. De lo contrario, es Pareto ineficiente.*

PALABRAS CLAVES: *liga deportiva; juegos en el campo; maximización de victorias; expectativa Pitagórica; nómina del equipo; eficiencia de Pareto.*

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1. Introduction

It is no exaggeration to say that among various things concerned with professional sports leagues, games played on the field attract fan interest most. Fans pay to watch games at the stadium, watch them on TV, read articles on them in newspapers and magazines, and talk passionately about them. In addition, some fans enjoy analyzing game statistics (on-field performance) of their favorite players and/or teams.

Nevertheless, in the economic analysis of professional sports leagues, games on the field have not been considered explicitly. For example, professional sports leagues have been studied by many economists: leagues with profit-maximizing teams (e.g., El-Hodiri & Quirk, 1971; Fort & Quirk, 1995; Vrooman, 1995; Szymanski, 2003; Szymanski & Késenne, 2004; Késenne, 2007; Madden, 2011); leagues with win-maximizing teams (e.g., Késenne, 2000a, 2006, and 2007; Vrooman, 2007); and leagues with utility-maximizing teams (e.g., Quirk & El-Hodiri, 1974; Rascher, 1997; Dietl, Grossman, & Lang, 2011). In these studies, phenomena that determine the results of games have been treated as events occurring inside a "black box", and the win percentage of a team has been determined by an exogenously given contest success function (CSF).

The purpose of this article is to incorporate games on the field into the economic analysis of sporting contests. For the purpose, we study professional sports leagues where (i) each team consists of offensive and defensive units, and plays a sequence of Attack & Defense games against each of the other teams during the league's regular season; (ii) the expected season win percentage of each team is calculated based on the expected points scored (EPS) and the expected points allowed (EPA) in the regular-season games by use of the Pythagorean expectation; and (iii) the offensive and defensive talent levels of each team, which are measured in terms of a point in a game, are determined so as to maximize the expected season win percentage subject to the budget constraint, given the talent levels of the opponents.

In these settings, it is shown that if the payroll of each team is different, then the teams in the league are divided into rich offensive-minded winners and poor defensive-minded losers, and the unique equilibrium talent allocation is Pareto inefficient in that there remain the possibilities of Pareto-improving player trades in the player market. In contrast, if the payroll of each team is the same, then all teams choose the same talent levels and the unique equilibrium talent allocation is Pareto efficient.

Here we mention the two analytical tools used in this article. First, the Attack & Defense game is a member of the matching penny family (i.e., zero-sum games with a unique mixed strategy Nash equilibrium) which is widely used to study strategic interactions in sports (e.g., Chiappori, Levitt, & Groseclose, 2002; Palacios-Huerta, 2003; Coloma, 2007; McGarrity & Linnen, 2010; Azar & Bar-Eli, 2011).[1] Our novelty is that the payoffs to players in the payoff matrix are not constants but variables that depend on other strategic variables previously chosen by the players. Second, the Pythagorean expectation is the formula that provides the win percentage a sports team should be expected to have at a particular time during a season. It was initially devised by the baseball statistician Bill James during the early 1980s, and has since been updated for use in other sports: American football (Schatz, 2003), basketball (Oliver, 2004; Chen & Tengfei, 2016), hockey (Cochran & Blackstock, 2009; Dayaratna & Miller, 2013), and soccer (Hamilton, 2011).

The remainder of this article is organized as follows. Section 2 introduces the Attack & Defense game and the Pythagorean expectation into a two-team league. Section 3 identifies an equilibrium allocation of the offensive and defensive talent between the two teams. Section 4 examines geometrically the results obtained in the preceding section by use of the box diagram, which is analogous to the Edgeworth box. Section 5 discusses a general n-team league and presents the main results of this article. Section 6 summarizes the key findings and concludes.

2. The Model

the Attack & Defense Game: Games on the Field

We consider a two-team league where each team consists of an offensive unit and a defensive unit.[2] The offensive talent level and the defensive talent level of team i are denoted by $x_i > 0$ and $y_i > 0$, respectively. A unit measuring x_i (y_i) is determined in such a way that a one-unit increase in x_i (y_i) increases (decreases) points scored (allowed) by one in a game played on the field.[3] As a result, we can express the expected points scored (EPS) and the expected points allowed (EPA) in regular-season games as functions of the talent levels. The determination of the talent levels will be discussed in the next section.

During the league's regular season, each team plays a sequence of Attack & Defense games against each other. In this game, the objective of the offense

[1] In addition, Dobson and Goddard (2010) studied the game where each team chooses between defensive and attacking formations and between non-violent and violent playing styles, which is a member of the family of non-zero-sum games with a unique dominant strategy equilibrium. In their model, the game is played finitely many times and the dominant strategy in a subgame at one stage of a match varies according to the situation (exogenously given parameter values), e.g. relative team quality, match duration, and difference in scores. Since their game always has a unique dominant strategy, they study "strategic behavior", but not "strategic interaction" (, which we study).

[2] Members of the offensive and defensive units can be different (e.g., American football) or the same (e.g., baseball, basketball, hockey, rugby, soccer, and volleyball).

[3] In this context, sabermetricians use various statistics to quantify a baseball player's entire offensive (defensive) performance by attempting to measure how many runs a player contributed (saved), e.g., Runs Created and Weighted Runs Above Average (offense), and Defensive Runs Saved and Ultimate Zone Rating (defense). See, for example, Glossary of the MLB website.

(Attack) is to score as many points as possible, while the objective of the defense (Defense) is to prevent the opposing offense from scoring. To do so, the offense tries to choose the type of play against which the defense does not defend, while the defense tries to defend against the type of play which the offense chooses. For simplicity, the offense and the defense have two alternative types of play, respectively (e.g., a type of play: run or pass, or a point of attack: short or deep, inside or outside, right or left, high or low, or timing: quick or delayed, etc.), and independently and simultaneously choose one of them.

The outcomes of the Attack & Defense game G_{ij} $(i \neq j)$ where team i is the offense (row player) and team j is the defense (column player) are shown in the payoff matrix depicted in Figure 1 (for the outcomes of the game G_{ji} where team i is the defense (column player) and team j is the offense (row player), change the subscripts i and j in each entry of the bi-matrix).[4] We define a sequence of the games $G_{\{i,j\}} \equiv \{G_{ij}, \cdots, G_{ij}, G_{ji}, \cdots, G_{ji}, G_{ij}, \cdots, G_{ij}, \cdots\}$ as a regular-season game played by teams i and j.

	D_1	D_2
A_1	$x_i - y_j, y_j - x_i$	$x_i, -x_i$
A_2	$x_i, -x_i$	$x_i - y_j, y_j - x_i$

Figure 1. The Attack & Defense game G_{ij} $(i \neq j)$

The entries in boxes (A_1, D_2) and (A_2, D_1) indicate that if the offense and the defense choose the different types of play, then the defense is unproductive, and thus the offense scores x_i points and the defense allows $|-x_i|$ points. On the other hand, the entries in boxes (A_1, D_1) and (A_2, D_2) indicate that if the offense and the defense choose the same types of play, then the defense is productive, and thus the offense scores $x_i - y_j$ points and the defense allows $|y_j - x_i|$ points.

In the remainder of this subsection, we derive the EPS and the EPA of a team in the regular-season game $G_{\{i,j\}}$. To do so, we decompose $G_{\{i,j\}}$ into two sequences of games $\{G_{ij}\}$ and $\{G_{ji}\}$, and calculate a Nash equilibrium of G_{ij} (we can also calculate a Nash equilibrium of G_{ji} in the same way). Since $x_i >$

[4] In ball games, e.g. American football, basketball, hockey, rugby, soccer, and volleyball, we can consider the team that is in possession of the ball to be the offense and the team that is not to be the defense.

$x_i - y_j$ for any $x_i > 0$ and $y_j > 0$, this zero-sum game has no pure strategy Nash equilibrium. We thus introduce a mixed strategy in which the offense chooses A_1 with probability p and A_2 with probability $1-p$ $(0 < p < 1)$, and a mixed strategy in which the defense chooses D_1 with probability q and D_2 with probability $1-q$ $(0 < q < 1)$.[5] By the payoff-equating method, solving the following equations:[6]

$$p(x_i - y_j) + (1-p)x_i = px_i + (1-p)(x_i - y_j),$$

$$q(x_i - y_j) + (1-q)x_i = qx_i + (1-q)(x_i - y_j),$$

we obtain the unique mixed strategy Nash equilibrium of the game $(p,q) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

The EPS and the EPA of team i in the regular-season game $G_{\{i,j\}}$ against team j are then given by

$$EPS_i^j = \frac{1}{4}(x_i - y_j + x_i + x_i + x_i - y_j) = \frac{1}{2}(2x_i - y_j), \quad (1)$$

$$EPA_i^j = \frac{1}{4}(x_j - y_i + x_j + x_j + x_j - y_i) = \frac{1}{2}(2x_j - y_i), \quad (2)$$

where $EPA_i^j = EPS_j^i$ (since points allowed by team i are points scored by team j, and vice versa). Equations (1) and (2) imply that a one-unit increase in x_i increases the EPS by one and a two-unit increase in y_i decreases the EPA by one.

Two remarks are in order.

Remark 1. Since the Attack & Defense game is a two-person zero-sum game with a unique mixed strategy Nash equilibrium, the use of Equations (1) and (2) (the values of G_{ij} and G_{ji}) as the EPS and the EPA of the regular-season game $G_{\{i,j\}} \equiv \langle \{G_{ij}\}, \{G_{ji}\} \rangle$ can be justified by the following result: Robinson (1951) proved that if the players of a finite two-person zero-sum game interact repeatedly via fictitious play, then the weighted averages of per-period payoffs converge to the value of the game.[7] If the Attack & Defense game G_{ij} is played

[5] The concept of mixed strategy has been used by many researchers to study strategic interactions in sports: e.g., American football (Kovash & Levitt, 2009; McGarrity & Linnen, 2010), baseball (Kovash & Levitt, 2009; Weinstein-Gould, 2009; Gmeiner, 2019), and soccer (Chiappori, Levitt, & Groseclose, 2002; Palacios-Huerta, 2003; Coloma, 2007; Azar & Bar-Eli, 2011).

[6] A mixed strategy Nash equilibrium has the property that if a player is playing his equilibrium mixed strategy, the expected payoffs to another player from playing her pure strategies must be identical. See, for example, Varian (1992, Section 15.5).

[7] The fictitious play was originally proposed by Brown (1951) as an algorithm for finding the approximate value of a two-person zero-sum game, and was later interpreted as a learning process for boundedly rational players in which each player plays a myopic pure best response in each period, on the assumption that the opponents will play each pure strategy with probability equal to its historical frequency. The method of forming an assessment over the

repeatedly, then for any initial weights satisfying $\kappa_0^i(D_1) \neq \kappa_0^i(D_2)$ and $\kappa_0^j(A_1) \neq \kappa_0^j(A_2)$ the fictitious play process generates a deterministic cycle of play $\cdots \to (A_1, D_1) \to \cdots \to (A_2, D_1) \to \cdots \to (A_2, D_2) \to \cdots \to (A_1, D_2) \to \cdots \to (A_1, D_1) \to \cdots$.

Remark 2. In general cases where the offense and the defense have the finite number $m \geq 3$ of alternative types of play, the EPS and the EPA of team i in the regular-season game against team j are given by

$EPS_i^j = \frac{1}{m}(mx_i - y_j)$ and $EPA_i^j = \frac{1}{m}(mx_j - y_i)$.

This generalization does not change the results of the model in any essential ways.

The following example shows that in general cases the fictitious play process generates more complicated cycles of play.

Example.

We consider the 3×3 Attack and Defense game G_{ij} where the payoffs when team i chooses A_a ($a = 1,2,3$) and team j chooses D_d ($d = 1,2,3$) are $(x_i - y_j, y_j - x_i)$ if $a = d$ and $(x_i, -x_i)$ if $a \neq d$, respectively. If G_{ij} is played repeatedly, then for given initial weights satisfying $\kappa_0^i(D_1) \neq \kappa_0^i(D_2) \neq \kappa_0^i(D_3)$ and $\kappa_0^j(A_1) \neq \kappa_0^j(A_2) \neq \kappa_0^j(A_3)$ the fictitious play process generates either of the following cycles of play (when the teams have initial weights "*" the cycle of play starts at "$(A_a, D_d)^*$", and so on):

Case (i) * $\kappa_0^i(D_1) < \kappa_0^i(D_2) < \kappa_0^i(D_3)$ and $\kappa_0^j(A_2) < \kappa_0^j(A_3) < \kappa_0^j(A_1)$,

** $\kappa_0^i(D_2) < \kappa_0^i(D_3) < \kappa_0^i(D_1)$ and $\kappa_0^j(A_3) < \kappa_0^j(A_1) < \kappa_0^j(A_2)$,

*** $\kappa_0^i(D_3) < \kappa_0^i(D_1) < \kappa_0^i(D_2)$ and $\kappa_0^j(A_1) < \kappa_0^j(A_2) < \kappa_0^j(A_3)$.

Figure-0 cycle:

$\cdots \to (A_1, D_1)^* \to \cdots \to (A_2, D_1) \to \cdots \to (A_2, D_2)^{**} \to \cdots \to (A_3, D_2) \to \cdots \to (A_3, D_3)^{***} \to \cdots \to (A_1, D_3) \to \cdots \to (A_1, D_1)^* \to \cdots$;

Case (ii) * $\kappa_0^i(D_1) < \kappa_0^i(D_3) < \kappa_0^i(D_2)$ and $\kappa_0^j(A_3) < \kappa_0^j(A_2) < \kappa_0^j(A_1)$,

** $\kappa_0^i(D_3) < \kappa_0^i(D_2) < \kappa_0^i(D_1)$ and $\kappa_0^j(A_2) < \kappa_0^j(A_1) < \kappa_0^j(A_3)$,

distribution of opponents' actions corresponds to Bayesian inference when a player believes that his opponents' play corresponds to a sequence of i.i.d. multinomial random variables with a fixed but unknown distribution, and a player's prior beliefs over this unknown distribution have a Dirichlet distribution. The fictitious play is foundation for more complex models (see Fudenberg & Levine, 1998, Chapter 2).

*** $\kappa_0^i(D_2) < \kappa_0^i(D_1) < \kappa_0^i(D_3)$ and $\kappa_0^j(A_1) < \kappa_0^j(A_3) < \kappa_0^j(A_2)$.

Figure-0 cycle:

$\cdots \rightarrow (A_1, D_1)^* \rightarrow \cdots \rightarrow (A_3, D_1) \rightarrow \cdots \rightarrow (A_3, D_3)^{**} \rightarrow \cdots \rightarrow (A_2, D_3) \rightarrow \cdots \rightarrow (A_2, D_2)^{***} \rightarrow \cdots \rightarrow (A_1, D_2) \rightarrow \cdots \rightarrow (A_1, D_1)^* \rightarrow \cdots$;

Case (iii) * $\kappa_0^i(D_1) < \kappa_0^i(D_2) < \kappa_0^i(D_3)$ and $\kappa_0^j(A_3) < \kappa_0^j(A_2) < \kappa_0^j(A_1)$,

** $\kappa_0^i(D_1) < \kappa_0^i(D_3) < \kappa_0^i(D_2)$ and $\kappa_0^j(A_2) < \kappa_0^j(A_3) < \kappa_0^j(A_1)$.

Figure-8 cycle:

$\cdots \rightarrow (A_1, D_1)^* \rightarrow \cdots \rightarrow (A_2, D_1) \rightarrow \cdots \rightarrow (A_2, D_2) \rightarrow \cdots \rightarrow (A_3, D_2) \rightarrow \cdots\cdots \rightarrow (A_3, D_3) \rightarrow \cdots \rightarrow (A_1, D_3) \rightarrow \cdots \rightarrow (A_1, D_1)^{**} \rightarrow \cdots \rightarrow (A_3, D_1) \rightarrow \cdots \rightarrow (A_3, D_3) \rightarrow \cdots \rightarrow (A_2, D_3) \rightarrow \cdots\cdots \rightarrow (A_2, D_2) \rightarrow \cdots \rightarrow (A_1, D_2) \rightarrow \cdots \rightarrow (A_1, D_1)^* \rightarrow \cdots$;

Case (iv) * $\kappa_0^i(D_2) < \kappa_0^i(D_3) < \kappa_0^i(D_1)$ and $\kappa_0^j(A_1) < \kappa_0^j(A_3) < \kappa_0^j(A_2)$,

** $\kappa_0^i(D_2) < \kappa_0^i(D_1) < \kappa_0^i(D_3)$ and $\kappa_0^j(A_3) < \kappa_0^j(A_1) < \kappa_0^j(A_2)$.

Figure-8 cycle:

$\cdots \rightarrow (A_2, D_2)^* \rightarrow \cdots \rightarrow (A_3, D_2) \rightarrow \cdots \rightarrow (A_3, D_3) \rightarrow \cdots \rightarrow (A_1, D_3) \rightarrow \cdots\cdots \rightarrow (A_1, D_1) \rightarrow \cdots \rightarrow (A_2, D_1) \rightarrow \cdots \rightarrow (A_2, D_2)^{**} \rightarrow \cdots \rightarrow (A_1, D_2) \rightarrow \cdots \rightarrow (A_1, D_1) \rightarrow \cdots \rightarrow (A_3, D_1) \rightarrow \cdots\cdots \rightarrow (A_3, D_3) \rightarrow \cdots \rightarrow (A_2, D_3) \rightarrow \cdots \rightarrow (A_2, D_2)^* \rightarrow \cdots$;

Case (v) * $\kappa_0^i(D_3) < \kappa_0^i(D_1) < \kappa_0^i(D_2)$ and $\kappa_0^j(A_2) < \kappa_0^j(A_1) < \kappa_0^j(A_3)$,

** $\kappa_0^i(D_3) < \kappa_0^i(D_2) < \kappa_0^i(D_1)$ and $\kappa_0^j(A_1) < \kappa_0^j(A_2) < \kappa_0^j(A_3)$.

Figure-8 cycle:

$\cdots \rightarrow (A_3, D_3)^* \rightarrow \cdots \rightarrow (A_1, D_3) \rightarrow \cdots \rightarrow (A_1, D_1) \rightarrow \cdots \rightarrow (A_2, D_1) \rightarrow \cdots\cdots \rightarrow (A_2, D_2) \rightarrow \cdots \rightarrow (A_3, D_2) \rightarrow \cdots \rightarrow (A_3, D_3)^{**} \rightarrow \cdots \rightarrow (A_2, D_3) \rightarrow \cdots \rightarrow (A_2, D_2) \rightarrow \cdots \rightarrow (A_1, D_2) \rightarrow \cdots\cdots \rightarrow (A_1, D_1) \rightarrow \cdots \rightarrow (A_3, D_1) \rightarrow \cdots \rightarrow (A_3, D_3)^* \rightarrow$,

where the length of period "$\cdots\cdots$" is longer than that of "$\cdots$".

The fictitious play process cycles as follows. Case (i): If the initial weights are "*", then team i chooses A_1 and team j chooses D_1 repeatedly, so the weights $\kappa_t^i(D_1)$ and $\kappa_t^j(A_1)$ increase. When team i's weights are $\kappa_t^i(D_2) < \kappa_t^i(D_1)$ team i switches to A_2 and chooses it repeatedly, so the weight $\kappa_t^j(A_2)$ increases. If the weights satisfy "**", then team i chooses A_2 and team j chooses D_2 repeatedly, and so on. The other results can be verified in similar ways.

The Expected Win Percentage: the Pythagorean Expectation

We specify the relationship between the season win percentage and the talent levels of a team. The expected season win percentage of team i is calculated based on Equations (1) and (2) by use of the Pythagorean expectation:

$$w_i = \frac{(2x_i - y_j)^\gamma}{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma}, \quad (3)$$

where $\gamma > 0$ is the Pythagorean exponent, $2x_i - y_j \geq 0$, and $2x_j - y_i \geq 0$. The values of Equation (3) for any $\gamma > 0$ are presented as follows (see Figure 2):

(i) if $x_i = y_j/2$ and $y_i < 2x_j$ (i.e., $EPS_i^j = 0$ and $EPA_i^j > 0$), then $w_i = 0$,

(ii) if $2(x_i - y_j/2) \lesseqgtr 2x_j - y_i$ (i.e., $EPS_i^j \lesseqgtr EPA_i^j$), then $w_i \lesseqgtr \frac{1}{2}$,

(iii) if $x_i > y_j/2$ and $y_i = 2x_j$ (i.e., $EPS_i^j > 0$ and $EPA_i^j = 0$), then $w_i = 1$.

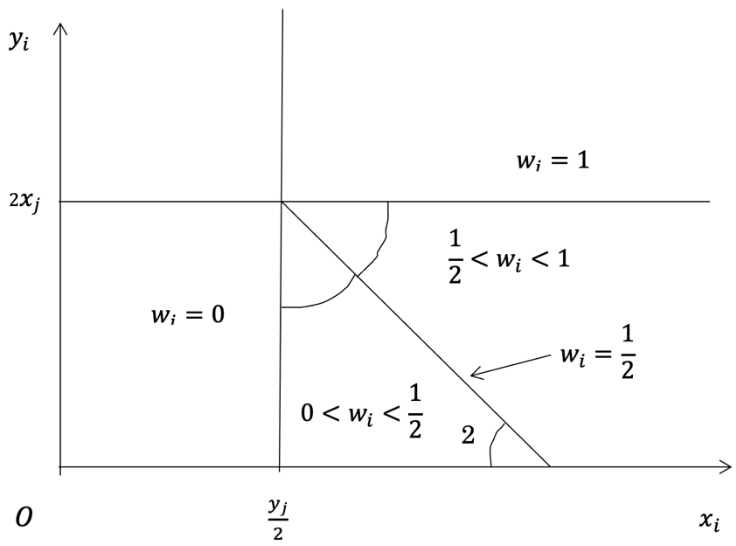

Figure 2. The level sets of the win percentage of team i

Thus, the level set of Equation (3) is a straight line segment through the point $(x_i, y_i) = (y_j/2, 2x_j)$ for given (x_j, y_j) $(j \neq i)$.

In the remainder of this subsection, we provide the relationship between Equation (3) and the contest success function (CSF) $w_i = x_i^\gamma/(x_i^\gamma + x_j^\gamma)$ with $0 < \gamma \leq 1$, which is the most widely used in the sports economics literature. The following properties of Equation (3) are analogous to those of the CSF mentioned above.[8] First, Equation (3) is non-negative, $w_i \geq 0$ and $w_j \geq 0$.

[8] As shown by Skaperdas (1996, Theorem 2), the power form is the only functional form that satisfies positivity and

Second, since $EPA_i^j = EPS_j^i$, Equation (3) satisfies the adding-up constraint $w_i + w_j = 1$ for any (x_i, y_i) and (x_j, y_j).[9] Third, Equation (3) is homogeneous of degree 0:

$$w_i = \frac{(2x_i - y_j)^\gamma}{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma} = \frac{(2\lambda x_i - \lambda y_j)^\gamma}{(2\lambda x_i - \lambda y_j)^\gamma + (2\lambda x_j - \lambda y_i)^\gamma}$$

for any $\lambda > 0$ and for any (x_i, y_i) and (x_j, y_j). Fourth, if we assume that $2x_i - y_j > 0$ and $2x_j - y_i > 0$, then Equation (3) is increasing in the talent levels of the team:

$$\frac{\partial w_i}{\partial x_i} = \frac{2\gamma(2x_i - y_j)^{\gamma-1}(2x_j - y_i)^\gamma}{\left\{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma\right\}^2} > 0,$$

$$\frac{\partial w_i}{\partial y_i} = \frac{\gamma(2x_i - y_j)^{\gamma}(2x_j - y_i)^{\gamma-1}}{\left\{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma\right\}^2} > 0,$$

and decreasing in the talent levels of the opponent:

$$\frac{\partial w_i}{\partial x_j} = -\frac{2\gamma(2x_i - y_j)^{\gamma}(2x_j - y_i)^{\gamma-1}}{\left\{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma\right\}^2} < 0,$$

$$\frac{\partial w_i}{\partial y_j} = -\frac{\gamma(2x_i - y_j)^{\gamma-1}(2x_j - y_i)^{\gamma}}{\left\{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma\right\}^2} < 0.$$

Lastly, Equation (3) has an anonymity property: $w_i = w_j = 1/2$ if $(x_i, y_i) = (x_j, y_j)$.

Since the Pythagorean exponent $\gamma > 0$ is usually greater than one, the shape of Equation (3) is more complicated than that of the CSF mentioned above.[10] The second-order partial derivatives of Equation (3) are given by

$$\frac{\partial^2 w_i}{\partial x_i^2} = -\frac{4\gamma(2x_i - y_j)^{\gamma-2}(2x_j - y_i)^\gamma\left\{(\gamma+1)(2x_i - y_j)^\gamma - (\gamma-1)(2x_j - y_i)^\gamma\right\}}{\left\{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma\right\}^3},$$

$$\frac{\partial^2 w_i}{\partial y_i^2} = \frac{\gamma(2x_i - y_j)^{\gamma}(2x_j - y_i)^{\gamma-2}\left\{(\gamma+1)(2x_j - y_i)^\gamma - (\gamma-1)(2x_i - y_j)^\gamma\right\}}{\left\{(2x_i - y_j)^\gamma + (2x_j - y_i)^\gamma\right\}^3},$$

the adding-up constraint, homogeneity, monotonicity, anonymity, consistency, and independence from irrelevant alternatives.

[9] Let n be the number of teams and n_i be the number of wins of team i. Then we have $w_i = n_i/(n-1)$. Since the total number of wins is $\sum n_i = n(n-1)/2$, we obtain $\sum w_i = n/2$.

[10] In James's original Pythagorean formula for baseball, the exponent was $\gamma = 2$. The optimal value of γ for baseball was later calculated to be 1.82-1.86 (Miller, 2007; Tung, 2010). For American football, an appropriate value for γ is 2.37 (Schatz, 2003). For basketball, an appropriate value for γ is around 14 (Oliver, 2004; Chen & Tengfei, 2016). For hockey, an appropriate value for γ is around 2 (Cochran & Blackstock, 2009; Dayaratna & Miller, 2013). For soccer, an appropriate value for γ is 1.60-1.85 (Hamilton, 2011).

$$\frac{\partial^2 w_i}{\partial x_i \partial y_i} = \frac{2\gamma^2 (2x_i - y_j)^{\gamma-1} (2x_j - y_i)^{\gamma-1} \left\{ (2x_j - y_i)^{\gamma} - (2x_i - y_j)^{\gamma} \right\}}{\left\{ (2x_i - y_j)^{\gamma} + (2x_j - y_i)^{\gamma} \right\}^3}.$$

Consequently, we obtain the following results:

(i) if $w_i < \frac{\gamma-1}{2\gamma}$, then $\frac{\partial^2 w_i}{\partial x_i^2} > 0$, $\frac{\partial^2 w_i}{\partial y_i^2} > 0$, and $\frac{\partial^2 w_i}{\partial x_i \partial y_i} > 0$,

(ii) if $\frac{\gamma-1}{2\gamma} < w_i < \frac{1}{2}$, then $\frac{\partial^2 w_i}{\partial x_i^2} < 0$, $\frac{\partial^2 w_i}{\partial y_i^2} > 0$, and $\frac{\partial^2 w_i}{\partial x_i \partial y_i} > 0$,

(iii) if $\frac{1}{2} < w_i < \frac{\gamma+1}{2\gamma}$, then $\frac{\partial^2 w_i}{\partial x_i^2} < 0$, $\frac{\partial^2 w_i}{\partial y_i^2} > 0$, and $\frac{\partial^2 w_i}{\partial x_i \partial y_i} < 0$,

(iv) if $\frac{\gamma+1}{2\gamma} < w_i$, then $\frac{\partial^2 w_i}{\partial x_i^2} < 0$, $\frac{\partial^2 w_i}{\partial y_i^2} < 0$, and $\frac{\partial^2 w_i}{\partial x_i \partial y_i} < 0$,

where $(\gamma - 1)/2\gamma > 0$ and $(\gamma + 1)/2\gamma < 1$ if $\gamma > 1$, while $(\gamma - 1)/2\gamma \leq 0$ and $(\gamma + 1)/2\gamma \geq 1$ if $0 < \gamma \leq 1$. For example, if $\gamma = 2$, then we have (i) $0 < w_i < 1/4$, (ii) $1/4 < w_i < 1/2$, (iii) $1/2 < w_i < 3/4$, and (iv) $3/4 < w_i < 1$.

The more significant difference is that in the CSF team i cannot choose $x_j = 0$ $(j \neq i)$, whereas in Equation (3) it can choose $y_i > 0$ so that $EPA_i^j = 0$ for given $x_j > 0$.

3. Determination of an Allocation of Talent: Maximization of the Expected Win Percentage subject to the Budget Constraint

In this section, we will identify an equilibrium allocation of the offensive and defensive talent between the two teams $E = \left((x_i, y_i), (x_j, y_j) \right)$. To do so, we consider win-maximizing teams. Before the league's regular season begins, each team independently determines the offensive talent level demanded $x_i > 0$ and the defensive talent level demanded $y_i > 0$ so as to maximize the expected season win percentage (Equation (3)) subject to the budget constraint, given the talent levels of the opponent (x_j, y_j).

Let $c_x > 0$ and $c_y > 0$ be the prices per unit of offensive and defensive talent, respectively. Then the budget constraints of the teams can be written as

$$c_x x_i + c_y y_i = M, \quad (4)$$

$$c_x x_j + c_y y_j = M. \quad (5)$$

We assume that the fixed amount of money available $M > 0$ is the same for each team. As we will see later, this assumption can be dropped in the $n \geq 3$-team case.

The best responses of win-maximizing teams to each other's choices of talent are determined so that the budget constraint (Equations (4) and (5), respectively) be satisfied and the EPA (Equation (2)) be zero:

$$x_i = M/c_x - 2(c_y/c_x)x_j \text{ and } y_i = 2x_j \text{ for } 0 < x_j < M/c_x, \tag{6}$$

$$x_j = M/c_x - 2(c_y/c_x)x_i \text{ and } y_j = 2x_i \text{ for } 0 < x_i < M/c_x. \tag{7}$$

Equations (6) and (7) imply that both win-maximizing teams buy as much offensive talent as possible under the budget constraint, and strengthen the defense in trying to win the game played against each other. Consequently, $EPS_j^i = 0$ and $EPS_i^j = 0$, and thus $w_i = w_j = 1/2$ (draw); that is, all undefeated teams won no games (a tragedy of undefeated teams).

The common price ratio (the exchange rate of talent) in Equations (6) and (7) is determined so that the marginal rate of substitution (MRS) between the offensive and defensive talent for each team be equal to it:

$$\frac{c_x}{c_y} = MRS_i = MRS_j, \text{ where } MRS_i \equiv \frac{\partial w_i/\partial x_i}{\partial w_i/\partial y_i}.$$

If this condition is satisfied at the talent allocation E, then E is Pareto efficient.[11] Since $w_i = w_j = 1/2$, we obtain $MRS_i = MRS_j = 2$ and thus $c_x = 2c_y$ (see Figure 2). The result that $MRS_i = 2$ follows from the result that a one-unit decrease in x_i decreases Equation (1) by one and a two-unit increase in y_i decreases Equation (2) by one. Thus, the Pareto efficiency condition $c_x = 2c_y$ implies that the cost of increasing the EPS equals the cost of decreasing the EPA.

Here we identify the Pareto efficient equilibrium talent allocation between the teams. Substituting $c_x = 2c_y$ into Equation (6) or (7), we obtain

$$x_i + x_j = \frac{M}{c_x}. \tag{8}$$

Substituting Equation (5) and $c_x = 2c_y$ into Equation (6), or substituting Equation (4) and $c_x = 2c_y$ into Equation (7), we obtain

$$y_i + y_j = \frac{M}{c_y}. \tag{9}$$

By Equations (6) and (7), $x_i \gtreqless x_j \Leftrightarrow y_i \lesseqgtr y_j$. Equations (8) and (9) together imply that in the two-team case the Pareto efficient equilibrium talent allocation E is not unique. This indeterminacy of equilibrium allocation will be explored further in the next section. As we will see later, in the $n \geq 3$-team case the Pareto efficient equilibrium talent allocation is unique.

[11] This condition is analogous to a tangency condition discussed in the context of the Edgeworth box. See, for example, Varian (1992, Section 17.6).

4. Geometrical Examination of the Equilibrium Talent Allocation

The results of the previous section can be examined geometrically by use of the box diagram depicted in Figure 3, which is analogous to the Edgeworth box.[12]

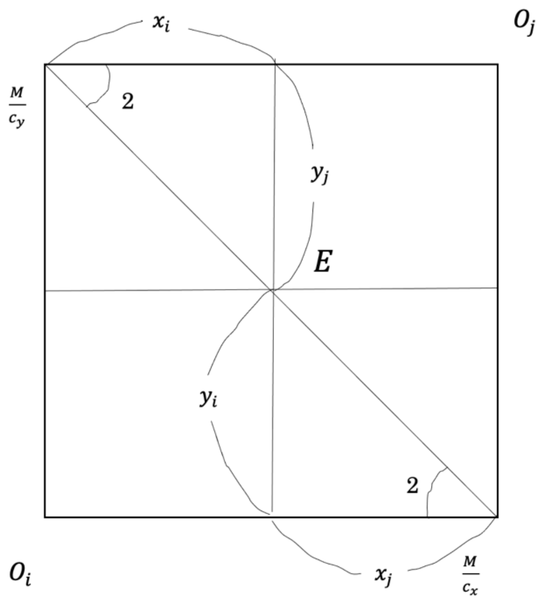

Figure 3. The box diagram for the two-team case

The sum of the offensive talent levels of the two teams (the length of the horizontal axis of the box) is given by Equation (8), and the sum of the defensive talent levels of the two teams (the height of the vertical axis of the box) is given by Equation (9). Geometrically, the talent levels of team i, (x_i, y_i), are measured from the lower left-hand corner O_i of the box, and the talent levels of team j, (x_j, y_j), are measured from the upper right-hand corner O_j of the box. In this way, every feasible allocation of the offensive and defensive talent between the two teams can be represented by a point in this box. The diagonal line with a slope of 2 is the budget lines, Equations (4) and (5), and the level sets $w_i = 1/2$ and $w_j = 1/2$ of the teams (for the level sets, see Figure 2). The budget set of team i (j) consists of all the points below and to the left (above and to the right) of this line. The point E is a unique intersection of the two level sets.

As shown in the previous section, by Equations (8) and (9), when the equilibrium talent allocation $E = \left((x_i, y_i), (x_j, y_j)\right)$ is Pareto efficient, team i chooses the talent levels $(x_i, y_i) = \left(M/c_x - x_j, M/c_y - y_j\right)$ for any given (x_j, y_j) that satisfies Equation (5). On the lower right-hand side of the box, where $x_i \geq y_j/2$ and $y_i \leq 2x_j$, the following results are depicted:

[12] For the Edgeworth box, see Varian (1992, Chapter 17) and Mas-Colell, Whinston, & Green (1995, Section 15.B).

$$2 = \frac{c_x}{c_y} = \frac{y_i}{M/c_x - x_i} = \frac{y_i}{x_j}. \tag{10}$$

The first equality follows from the Pareto efficiency condition, the second equality follows from Equation (4), and the last equality follows from Equation (8). By Equation (10), team i chooses a point (x_i, y_i) on the budget line so that $y_i = 2x_j$ for given (x_j, y_j) that lies on the budget line, as stated in Equation (6). The win percentage of team i attains its maximum value $1/2$ on the budget line.

Similarly, by Equations (8) and (9), team j chooses the talent levels $(x_j, y_j) = (M/c_x - x_i, M/c_y - y_i)$ for any given (x_i, y_i) that satisfies Equation (4). On the upper left-hand side of the box, where $x_j \geq y_i/2$ and $y_j \leq 2x_i$, the following results are depicted:

$$2 = \frac{c_x}{c_y} = \frac{y_j}{M/c_x - x_j} = \frac{y_j}{x_i}. \tag{11}$$

The first equality follows from the Pareto efficiency condition, the second equality follows from Equation (5), and the last equality follows from Equation (8). By Equation (11), team j chooses a point (x_j, y_j) on the budget line so that $y_j = 2x_i$ for given (x_i, y_i) that lies on the budget line, as stated in Equation (7). The win percentage of team j attains its maximum value $1/2$ on the budget line.

By Equations (4), (5), (10), and (11), the Pareto efficient equilibrium talent allocation E is determined. It can be verified that $x_i \gtreqless x_j \Leftrightarrow y_i \lesseqgtr y_j$. As stated in the previous section, a solution is not unique and any allocation on the budget lines can be a Nash equilibrium.

5. A General *n*-Team League

Asymmetric Contests

In the previous three sections, we have studied the two-team league. In this section, we will study a general n-team league. To do so, we consider an n-team league where each team consists of offensive and defensive units, and plays the game $G_{\{i,j\}} \equiv \langle \{G_{ij}\}, \{G_{ji}\} \rangle$ $(i \neq j)$ against each of the $n-1$ other teams during the league's regular season.

By an argument exactly analogous to that used in the two-team case in Section 2, the expected total points scored (EPS) and the expected total points allowed (EPA) of team i $(i = 1, \cdots, n)$ in the regular-season games are given by

$$EPS_i \equiv \sum_{j \neq i} EPS_i^j = \frac{n-1}{2}\left(2x_i - \frac{1}{n-1}\sum_{j \neq i} y_j\right), \tag{12}$$

$$EPA_i \equiv \sum_{j \neq i} EPA_i^j = \frac{n-1}{2}\left(\frac{2}{n-1}\sum_{j \neq i} x_j - y_i\right), \tag{13}$$

where $\sum EPS_i = \sum EPA_i$. Equations (12) and (13) imply that a one-unit increase in x_i increases the EPS by $n-1$ and a one-unit increase in y_i decreases the EPA by $(n-1)/2$. The expected season win percentage of team i is then calculated based on Equations (12) and (13) by use of the Pythagorean expectation:

$$w_i = \frac{\left(2x_i - \frac{1}{n-1}\sum_{j\neq i} y_j\right)^{\gamma}}{\left(2x_i - \frac{1}{n-1}\sum_{j\neq i} y_j\right)^{\gamma} + \left(\frac{2}{n-1}\sum_{j\neq i} x_j - y_i\right)^{\gamma}}, \qquad (14)$$

where $\gamma > 0$.

Here we identify the equilibrium offensive and defensive talent levels of the teams. Assume that the fixed amount of money available $M_i > 0$ is different for each team. Then the budget constraint of each team can be written as

$$c_x x_i + c_y y_i = M_i \text{ for all } i. \qquad (15)$$

By an argument exactly analogous to that used in the two-team case in Section 3, by Equation (14), the best response of each win-maximizing team to others' choices of talent is determined as

$$y_i = \frac{2}{n-1}\sum_{j\neq i} x_j \text{ for all } i. \qquad (16)$$

Substituting Equation (16) into Equation (15), we have

$$c_x x_i + \frac{2c_y}{n-1}\sum_{j\neq i} x_j = M_i. \qquad (17)$$

Summing Equation (17) over $i = 1, \cdots, n$ to get

$$\sum_{j\neq i} x_j = \frac{M_i + \sum_{j\neq i} M_j}{c_x + 2c_y} - x_i. \qquad (18)$$

Substituting Equation (18) back into Equation (17) and then solving for x_i, we obtain the equilibrium offensive talent level of each team:

$$x_i = \alpha_1 M_i - \alpha_2 \sum_{j\neq i} M_j \text{ for all } i, \qquad (19)$$

where $\alpha_1 = \frac{(n-1)c_x + 2(n-2)c_y}{\{(n-1)c_x - 2c_y\}(c_x + 2c_y)} > 0$ and $\alpha_2 = \frac{2c_y}{\{(n-1)c_x - 2c_y\}(c_x + 2c_y)} > 0$.

Substituting Equation (19) back into Equation (18) and then substituting the result into Equation (16), we obtain the equilibrium defensive talent level of each team:

$$y_i = -\beta_1 M_i + \beta_2 \sum_{j\neq i} M_j \text{ for all } i, \qquad (20)$$

where $\beta_1 = \frac{4c_y}{\{(n-1)c_x - 2c_y\}(c_x + 2c_y)} > 0$ and $\beta_2 = \frac{2c_x}{\{(n-1)c_x - 2c_y\}(c_x + 2c_y)} > 0$.

The equilibrium talent levels (Equations (19) and (20)) are positive if

$$\frac{n\alpha_2}{\alpha_1+\alpha_2}\cdot\frac{1}{n}\sum M_j < M_i < \frac{n\beta_2}{\beta_1+\beta_2}\cdot\frac{1}{n}\sum M_j,$$

where $n\alpha_2/(\alpha_1+\alpha_2)<1$ and $1<n\beta_2/(\beta_1+\beta_2)$ if $(n-1)c_x-2c_y>0$.

In the remainder of this subsection, we identify the effects of the gap in team payrolls. First, we examine the effects on the talent levels of the teams and show that the teams in the league are divided into the rich offensive-minded teams and the poor defensive-minded teams. It can be derived from Equations (19) and (20) that $\partial x_i/\partial M_i>0$, $\partial y_j/\partial M_i>0$, $\partial x_j/\partial M_i<0$, and $\partial y_i/\partial M_i<0$ for $i\neq j$. These results in turn imply that when the amount of money available to team i increases, it increases its demand for offensive talent, and the other teams respond to it by increasing their demand for defensive talent and reducing their demand for offensive talent under the budget constraints, and thus it reduces its demand for defensive talent. Therefore, we obtain $x_i>x_j$ and $y_i<y_j$ for $M_i>M_j$. It can also be verified from Equations (19) and (20) that $x_i \gtreqless \sum x_j/n$ and $y_i \lesseqgtr \sum y_j/n \Leftrightarrow M_i \gtreqless \sum M_j/n$.

Next, we move to the effects on the expected season win percentages of the teams and show that the teams in the league are divided into the rich winners and the poor losers. Substituting Equations (16) and (19) into Equation (12) gives us

$$EPS_i=\frac{n(n-2)}{n-1}\left\{x_i-\frac{1}{n}\sum x_j\right\}=\frac{n(n-2)}{\{(n-1)c_x-2c_y\}}\left\{M_i-\frac{1}{n}\sum M_j\right\}.$$

Since $n\geq 3$, we obtain the relation:

$$EPS_i \gtreqless 0 \quad\Leftrightarrow\quad M_i \gtreqless \frac{1}{n}\sum M_j.$$

On the other hand, substituting Equation (16) into Equation (13), we obtain $EPA_i=0$ for all i. (Unlike the two-team case, $EPA_i=0$ for all i does not necessarily imply that $EPS_i=0$ for all i since $EPS_i\equiv\sum_{j\neq i}EPS_i^j$ whereas $EPA_i=\sum_{j\neq i}EPS_j^i$.) We now interpret $EPS_i^j<0$ (i.e., $2x_i<y_j$) as points scored by the defense of team j and include $-EPS_i^j>0$ in the calculation of EPA_i'.[13] Similarly, we interpret $EPA_i^j<0$ (i.e., $2x_j<y_i$) as points scored by the defense of team i and include $-EPA_i^j>0$ in the calculation of EPS_i'. Then, if the payroll of team i is greater than the mean levels of all teams, $M_i>\sum M_j/n$, then we obtain $EPS_i'>EPA_i'>0$, and thus $1/2<w_i<1$. In contrast, if $M_i<\sum M_j/n$, then we obtain $0<EPS_i'<EPA_i'$, and thus $0<w_i<1/2$. (The

[13] In American football, the defensive unit can score points, e.g., a fumble return touchdown, an interception return touchdown, and a safety. In soccer, we can think of goals scored from counter-attacks. In volleyball, we can think of block points.

definitions of the EPS and the EPA are modified as stated above. See **Example** below.)

Example.

We consider a 3-team case where $M_1 = 1$, $M_2 = 2$, and $M_3 = 4$. In this case, the offensive talent levels are $x_1 < x_2 < \sum x_j/3 < x_3$ and the defensive talent levels are $y_1 > y_2 > \sum y_j/3 > y_3$. Substituting Equations (16) and (19) into Equation (1) gives us

$$EPS_i^j = x_i - \frac{1}{n-1}\sum_{k\neq j} x_k = \frac{n-1}{\{(n-1)c_x - 2c_y\}}\left\{M_i - \frac{1}{n-1}\sum_{k\neq j} M_k\right\},$$

where the sign of the term on the right can be determined by

$$M_i - \frac{1}{n-1}\sum_{k\neq j} M_k \gtreqless \frac{1}{n-1}\left\{M_j - \frac{1}{n}\sum M_k\right\} \quad \Leftrightarrow \quad M_i \gtreqless \frac{1}{n}\sum M_k.$$

Since $(\sum_{k\neq 3} M_k)/2 < M_2 < (\sum_{k\neq 1} M_k)/2$, we obtain $EPS_2^1 < 0$ and $EPS_2^3 > 0$. Similarly, since $M_1 < (\sum_{k\neq 2} M_k)/2 < M_3$, we obtain $EPA_2^1 = EPS_1^2 < 0$ and $EPA_2^3 = EPS_3^2 > 0$. Since $M_2 < \sum M_j/3$, we obtain the relations:

$$EPS_2 = EPS_2^1 + EPS_2^3 < 0 \quad \Leftrightarrow \quad EPS_2^3 < -EPS_2^1,$$

$$EPA_2 = EPA_2^1 + EPA_2^3 = 0 \quad \Leftrightarrow \quad -EPA_2^1 = EPA_2^3.$$

Thus, $EPS_2 < EPA_2 \Leftrightarrow EPS_2{}' < EPA_2{}'$, where $EPS_2{}' = EPS_2^3 - EPA_2^1 > 0$ and $EPA_2{}' = EPA_2^3 - EPS_2^1 > 0$. Therefore, the expected win percentage of team 2 is $0 < w_2 < 1/2$. The other results can be verified in similar ways. □

Lastly, we check the Pareto efficiency of the unique equilibrium talent allocation $E = ((x_i, y_i)_{i=1}^n)$ given by Equations (19) and (20), and show that E is Pareto inefficient. By Equations (12), (13), and (14), the MRS for team i is given by $MRS_i = 2(EPA_i{}'/EPS_i{}')$. Thus, we obtain $MRS_i \gtreqless 2 \Leftrightarrow w_i \lesseqgtr 1/2$. Therefore, the MRS is not equalized across the teams at E, and thus there remain the possibilities of Pareto-improving player trades in the player market. Another characterization of Pareto efficiency is that any talent allocation that satisfies the adding-up constraint is Pareto efficient. Unlike the two-team case, Equation (14) does not necessarily satisfy the adding-up constraint $\sum w_i = n/2$. The only Pareto efficient talent allocation that satisfies both the equalization of the MRS and the adding-up constraint leads to $w_i = 1/2$ for all i.

Symmetric Contests

The foregoing is contrasted with the case of symmetric contests. We now assume that the fixed amount of money available is the same for each team, $M_i = M$ for all i. This assumption is satisfied when a salary cap is adopted.[14]

[14] For the economic analysis of a salary cap, see Fort & Quirk (1995), Vrooman (1995), Késenne (2000b, 2007), and

Under this assumption, the unique equilibrium talent levels (Equations (19) and (20)) of each team become

$$x_i = \frac{M}{c_x+2c_y} \text{ and } y_i = \frac{2M}{c_x+2c_y} \text{ for all } i. \quad (21)$$

Substituting Equation (21) into Equation (14), we obtain $w_i = 1/2$ for all i (a tragedy of undefeated teams). Thus, the Pareto efficiency condition is given by

$$\frac{c_x}{c_y} = MRS_i = MRS_j = 2 \text{ for all } i \neq j.$$

(Note that unlike the two-team case $(n-1)c_x - 2c_y \neq 0$ for $n \geq 3$ in the denominators and numerators of Equations (19) and (20).) Substituting $c_x = 2c_y$ into Equation (21), we can say that the budget of each team is equally allocated to the offensive and defensive talent:

$$c_x x_i = c_y y_i = \frac{M}{2} \text{ for all } i. \quad (22)$$

In the remainder of this subsection, we provide an interpretation of the n-team symmetric contests. Since Equation (21) is also a symmetric solution to Equations (15), (16), and

$$c_x \frac{1}{n-1}\sum_{j\neq i} x_j + c_y \frac{1}{n-1}\sum_{j\neq i} y_j = M, \quad (23)$$

$$\frac{1}{n-1}\sum_{j\neq i} y_j = 2x_i, \quad (24)$$

the strategic interactions between team i and all other teams can be examined

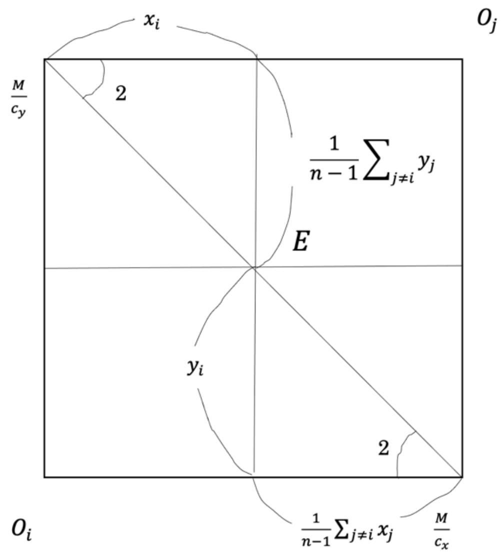

Figure 4. The box diagram for the n-team case

geometrically by use of the box diagram depicted in Figure 4, as in the two-team

Dietl et al. (2011).

case in Section 4. Equations (23) and (24) can be interpreted as the budget constraint and the best response of a fictitious "shadow" team, with the talent levels equal to the mean levels of all other teams.[15] So the strategic interaction environment faced by each team is analogous to the yardstick competition à la Shleifer (1985).[16]

Equation (24) can be obtained if a symmetric solution $\sum_{j \neq i} x_j/(n-1) = x_i$ is assumed. Substituting this symmetry assumption into Equation (16), we obtain $y_i = 2x_i$. Summing Equation (16) over $i = 1, \cdots, n$, and then substituting $y_i = 2x_i$ and the symmetry assumption into the result, we obtain Equation (24):

$$\frac{1}{n-1}\sum_{j \neq i} y_j = \frac{2}{n-1}\sum_{j \neq i} x_j = 2x_i,$$

which also implies that $\sum_{j \neq i} y_j/(n-1) = y_i$. These results can be shown graphically in Figure 4 (see the lower-right hand side ⇒ the lower-left hand side ⇒ the upper-right hand side ⇒ the upper-left hand side). By Equations (15) and (23) and the symmetry assumption, the Pareto efficient equilibrium talent allocation E is uniquely determined. It can be readily verified from the symmetry assumption that Equation (22) holds.

6. Concluding Remarks

In this article, we have incorporated games played on the field into the economic analysis of sporting contests. By doing so, we can systematically study the following three decision-making problems of each team: in chronological order, (i) each team (mainly the owner) determines the team payroll; (ii) each team (mainly the general manager) determines the offensive and defensive talent levels so as to maximize the expected season win percentage (Pythagorean expectation) subject to the budget constraint, given the talent levels of the opponents; and (iii) each team (mainly the manager) chooses the types of play for regular-season games so as to maximize the expected points scored (EPS) and minimize the expected points allowed (EPA), given the opponents' choices of play.

We have solved these problems backward and obtained the following results: for (iii) each team randomizes over alternative types of play in each game; for (ii) the talent levels of each team depend on the team payroll: the teams in the league are divided into the rich offensive-minded teams and the poor defensive-minded teams; and for (i) each team's performance on the field depends on the team payroll: if the payroll of a team is greater (smaller) than the mean levels of all teams, then the expected season win percentage of the

[15] The term "shadow" is taken from Shleifer (1985). In the yardstick competition, the shadow firm serves as the benchmark for a firm.

[16] Madden (2011, p.409) suggested that strategic interactions in sporting contests are analogous to the Cournot competition (with expenditures on talent replacing quantities of talent).

team is higher (lower) than 50%. Moreover, if the payroll of each team is the same, then the unique equilibrium talent allocation is Pareto efficient. Otherwise, it is Pareto inefficient. In the latter case, there remain the possibilities of Pareto-improving player trades in the player market.

For the result (ii), we can point out the following facts: in the 2019 Major League Baseball (MLB) season, Runs Created of eight of fourteen (ten of sixteen) teams that spent more (less) money on player salaries than the mean levels of all thirty teams were higher (lower) than the mean levels of all teams (the correlation coefficient between Runs Created and team payroll is 0.493).[17] On the other hand, Defensive Runs Saved of eleven of sixteen (six of fourteen) teams that spent less (more) money on player salaries than the mean levels of all thirty teams were non-negative (negative) (the correlation coefficient is 0.123).[18]

For the result (i), we can point out the following facts: First, in the 2019 MLB season, nine of fourteen (sixteen) teams that spent more (less) money on player salaries than the mean levels of all thirty teams won (lost) more than 50% of regular-season games (the correlation coefficient between team payroll and win percentages is 0.404).[19] Second, in the English Premier League (EPL) during the 2016-17 season, six teams (Manchester City, Manchester United, Chelsea, Arsenal, Liverpool, and Tottenham Hotspur) spent more money on player salaries than the mean levels of all twenty teams, and only seven teams (the above mentioned six teams in the top six and Everton in the seventh place) won more than 50% of regular-season games (the correlation coefficient is 0.864).[20]

According to the result (i), the strategic determination of team payroll is “a race to the top”, so a win maximizer will spend everything he can lay his hands on. An appropriate way of determining team payroll (e.g., profit maximization, a breakeven constraint, or salary restrictions (caps and floors), etc.) is a research subject for future studies. The effects of the upper level competition (e.g., the UEFA Champions League and the UEFA Europa League) on team payroll and those of promotion/relegation systems are also research subjects for future studies.

[17] The regression of Runs Created RC_i on team payroll M_i (million \$) is given by $RC_i = 673.83 + 1.022M_i$ and the t-value of the estimated slope is 2.995. Thus, the null hypothesis is rejected at the 1% level of significance. For Runs Created of the MBL teams in the 2019 season, see Baseball Reference website.

[18] In the regression of Defensive Runs Saved on team payroll, the t-value of the estimated slope is 0.654, and thus the null hypothesis cannot be rejected. For Defensive Runs Saved of the MLB teams in the 2019 season, see Fielding Bible website.

[19] The regression of win percentage w_i (%) on team payroll M_i (million \$) is given by $w_i = 38.72 + 0.085M_i$ and the t-value of the estimated slope is 2.335. Thus, the null hypothesis is rejected at the 5% level of significance. For the payroll data of the MBL teams in the 2019 season, see USA TODAY website.

[20] The regression of win percentage w_i (%) on team payroll M_i (million £) is given by $w_i = 28.991 + 0.217M_i$ and the t-value of the estimated slope is 7.273. Thus, the null hypothesis is rejected at the 1% level of significance. For the payroll data of the EPL teams during the 2016-17 season, see TOTAL SPORTEK (2018).

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MLB2019	Team	Payroll	RC	DRS	WP	
1	Cubs	211.546714	848	-14	51.9	ROW
2	Yankees	205.442088	944	-5	63.6	ROW
3	Nationals*	204.449127	907	-3	57.4	ROW
4	Red Sox	204.335019	941	-28	51.9	ROW
5	Giants	199.729652	677	47	47.5	
6	Angels	177.328583	794	9	44.4	
7	Dodgers	170.903333	930	126	65.4	
8	Astros	165.653	1004	96	66	
9	Cardinals	165.253599	764	91	56.2	
10	Mariners	152.807076	766	-88	42	
11	Rockies	150.805164	872	9	43.8	
12	Mets	149.00223	840	-86	53.1	ROW
13	Phillies	144.616127	799	51	50	
14	Reds	133.580714	747	58	46.3	
15	Brewers	127.850342	855	40	54.9	
16	Indians	122.875033	787	82	57.4	
17	Braves	117.855753	898	41	59.9	
18	Twins	114.901933	967	3	62.3	
19	Dbacks	108.04065	831	112	52.5	
20	Rangers	106.969999	799	-52	48.1	
21	Tigers	104.5819	644	-84	29.2	
22	Royals	102.570791	705	5	36.2	PDL
23	Athletics	102.545	830	36	59.9	
24	White Sox	90.652	730	-56	44.7	
25	Orioles	80.012045	737	-95	33.3	
26	Pirates	76.082999	782	-46	42.6	
27	Padres	75.795766	697	17	43.2	PDL
28	Blue Jays	71.228671	733	0	41.4	PDL
29	Marlins	70.61002	625	25	35.2	PDL
30	Rayes	53.500799	810	53	59.3	
	Average	132.0508709	808.76667		49.986667	

Team Payroll, Runs Created, Defensive Runs Saved, and Win Percentage in the 2019 MLB Season. (*: World Series Champion, ROW: Rich Offensive-minded Winner, PDL: Poor Defensive-minded Loser)